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Symmetric power sum expansions of the eigenvalues of generalised Casimir operators of semi-simple Lie groups

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Abstract. A formula due to Okubo for the eigenvalues of generalised Casimir operators of semi-simple Lie groups is used to derive an explicit expression for these eigenvalues. Full use is made of the Weyl symmetry group and it is shown that this expression may be cast in the form of a symmetric power sum expansion. Expansions are derived for operators of order p constructed using the defining representation of each simple Lie group for all $p \leqslant 8$. The results are in accord with the known facts regarding a complete set of algebraically independent operators and yield algebraic relations amongst those which are not independent. The expansions for the orthogonal and symplectic groups are a distinct improvement upon those obtained earlier, whilst those for the exceptional groups are the first of their kind.

1. Introduction

Casimir (1931) constructed an operator C_2 which commutes with all the generators of a compact semi-simple Lie group. This operator is of second order in the generators. Using the structure constants of the corresponding Lie algebra, Racah (1951) constructed invariant operators C_p of arbitrary order, p, in the generators. Gruber and O'Raifeartaigh (1964) made a further generalisation and defined invariants I_p^{μ} involving the characters of products of generators in irreducible representations, $\boldsymbol{\mu}$, of the Lie group. This generalisation is such that in the case for which $\boldsymbol{\mu}$ is the adjoint or regular representation, $\boldsymbol{\phi}$, I_p^{ϕ} is essentially C_p. A general expression for the eigenvalue $C_2(\boldsymbol{\lambda})$ of the second-order Casimir operator C_2 in any irreducible representation $\boldsymbol{\lambda}$ of a semi-simple Lie group was derived by Racah (1950). This formula expresses $C_2(\boldsymbol{\lambda})$ explicitly in terms of the highest weight vector labelling the irreducible representation, and the derivation made use of the canonical Cartan–Weyl basis for the Lie algebra. In the case of the classical Lie groups it has been found convenient to use a Gelfand–Okubo basis together with tensor operator methods. The work of Perelemov and Popov (1966a) led in the case of the unitary groups to an explicit formula for the eigenvalues of a set of generalised Casimir operators of arbitrary order. The simplest form of this result was given first by Popov and Perelemov (1967) and independently derived by Louck and Biedenharn (1970, p 2403) and Okubo (1975). In the case of the other classical groups the preliminary results of Perelemov and Popov (1966b) were followed up by Wong and Yeh (1975) and by Nwachuku and Rashid (1976) who gave

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explicit formulae for the eigenvalues of the Casimir operators of the orthogonal and symplectic groups. The result for the orthogonal group had previously been derived by Bracken and Green (1971). In all these cases the generalised Casimir operators were recognised by Popov and Perelemov (1968) to be nothing other than I_p^{ω} where $\boldsymbol{\omega}$ is the defining representation of the group. Through this recognition and an extension of their methods they were able to derive a preliminary result for the exceptional Lie group G_2.

An alternative approach to the problem was put forward by Agrawala and Belinfante (1971) who derived a very remarkable formula for the eigenvalues of generalised Casimir operators I_p^{ω} of the unitary group. This formula relates these eigenvalues directly to the eigenvalues of the second-order Casimir operator C_2. This result appears to have gone unnoticed, but Edwards (1978) recently presented a similar approach to the evaluation of the eigenvalues of Casimir operators of the classical groups which proceeded via intermediate formulae for $I_p^{\omega}(\boldsymbol{\lambda})$ and $I_p^{\bar{\omega}}(\boldsymbol{\lambda})$ which correspond exactly with the formulae of Agrawala and Belinfante. However even before this the required generalisation to the eigenvalues $I_p^{\mu}(\boldsymbol{\lambda})$ of the operators I_p^{μ} in the irreducible representation $\boldsymbol{\lambda}$ of any semi-simple Lie group had been given by Okubo (1977). It is this formula which is exploited in this paper.

Both Okubo (1977) and Edwards (1978) rederived very easily the results for the classical groups previously obtained by Popov and Perelemov (1967), Bracken and Green (1971) and Nwachuku and Rashid (1976). Okubo went further and applied his techniques to G_2, coming very close to the final simplification of the preliminary result for G_2 due to Popov and Perelemov (1968).

All of these results expressed the required eigenvalues $I_m^{\mu}(\boldsymbol{\lambda})$ rationally in terms of the components of the irreducible representation label $\boldsymbol{\lambda}$. In general the expression is in fact a sum of $d(\boldsymbol{\mu})$ rational terms where $d(\boldsymbol{\mu})$ is the dimension of the representation $\boldsymbol{\mu}$. This contrasts strikingly with the formula for $C_2(\boldsymbol{\lambda})$ derived by Racah (1950) which essentially expressed $C_2(\boldsymbol{\lambda})$ as the power sum $S_2(\boldsymbol{\lambda})$, i.e. as a second-degree multinomial in the components of $\boldsymbol{\lambda}$. Attempts have been made, notably by Umezawa (1963, 1964a, b) and Louck and Biedenharn (1970) to generalise this result of Racah's by expressing higher-order operator eigenvalues in terms of power sums $S_k(\boldsymbol{\lambda})$. This proved to be difficult, although some preliminary results were obtained including a result appropriate to G_2 calculated by Scheibling and Umezawa (1970). More recently Popov (1976) derived, using generating function methods, new expansions of the eigenvalues $I_p^{\omega}(\boldsymbol{\lambda})$ in terms of symmetric power sums in the case of the unitary groups. This work was extended to the orthogonal and symplectic groups by Nwachuku (1979) but the results obtained were not expressed in an ideal form, in that they did not make manifest relationships between eigenvalues of $I_p^{\omega}(\boldsymbol{\lambda})$ for various values of p. The existence of such relationships is implied by the fact that any rank-k semi-simple Lie group possesses at most k algebraically independent Casimir invariants. The orders of a complete set of independent invariants were listed by Racah (1950) for each simple Lie group. They are reproduced here in table 1.

In this paper the overwhelming power of the formula for $I_p^{\mu}(\boldsymbol{\lambda})$ due to Okubo (1977) is exploited, first to show that with an understanding of the modification rules appropriate to irreducible representations of groups Okubo's results for the classical groups can be extended without any difficulty to the case of all the exceptional groups. Secondly it is demonstrated that the resulting sum of rational terms may be re-expanded in terms of power sums $S_k(\boldsymbol{\lambda})$ and similar functions. Thirdly these expansions are obtained explicitly in the case of all operator eigenvalues $I_p^{\omega}(\boldsymbol{\lambda})$ with $p \leqslant 8$ for each simple Lie

Table 1. The orders of a complete set of algebraically independent invariants I_p^*.

Group	Order p
$A_k \sim SU(k+1)$	$2, 3, \ldots, k+1$
$B_k \sim SO(2k+1)$	$2, 4, \ldots, 2k$
$C_k \sim Sp(2k)$	$2, 4, \ldots, 2k$
$D_k \sim SO(2k)$	$2, 4, \ldots, 2k-2, k$
G_2	2, 6
F_4	2, 6, 8, 12
E_6	2, 5, 6, 8, 9, 12
E_7	2, 6, 8, 10, 12, 14, 18
E_8	2, 8, 12, 14, 18, 20, 24, 30

group. It is shown that these expansions exhibit the algebraic independence requirements given by Racah.

2. The Okubo formula

Let X_i with $i = 1, 2, \ldots, N$ be the generators of a Lie algebra of dimension N defined by

$$[X_i, X_j] = c_{ij}^{k} X_k \tag{2.1}$$

where the structure constants satisfy

$$c_{ij}^{k} = -c_{ji}^{k} \tag{2.2}$$

and

$$c_{ij}^{k} c_{kl}^{m} + c_{jl}^{k} c_{ki}^{m} + c_{li}^{k} c_{kj}^{m} = 0. \tag{2.3}$$

For such an algebra there exists a covariant, symmetric, tensor

$$g_{il} = c_{ij}^{k} c_{lk}^{j}. \tag{2.4}$$

If the Lie algebra is semi-simple this tensor is non-singular and can be used to define a contravariant, symmetric tensor such that

$$g_{il} g^{lm} = \delta_i^m. \tag{2.5}$$

This may be used to define

$$X^l = g^{lm} X_m. \tag{2.6}$$

The second-order Casimir operator then takes the form

$$C_2 = g^{im} X_i X_m = g_{il} X^i X^l = c_{ij}^{k} c_{lk}^{j} X^i X^l. \tag{2.7}$$

One generalisation of this operator due to Racah (1951) is

$$C_p = c_{i_1 j_1}^{j_2} c_{i_2 j_2}^{j_3} \ldots c_{i_p j_p}^{j_1} X^{i_1} X^{i_2} \ldots X^{i_p}. \tag{2.8}$$

The adjoint or regular representation $\boldsymbol{\phi}$ of the Lie algebra is defined by the map

$$\boldsymbol{\phi} : X_i \to D^{\phi}(X_i), \tag{2.9}$$

where the matrix elements in this representation are

$$D^{\phi}(X_i)_{kl} = -c_{ik}^{l}. \tag{2.10}$$

More generally any representation $\boldsymbol{\mu}$ of the Lie algebra is defined by a homomorphism onto a set of matrices:

$$\boldsymbol{\mu} : X_i \rightarrow D^{\mu}(X_i). \tag{2.11}$$

This affords the further generalisation of the Casimir operator given by Gruber and O'Raifeartaigh (1964):

$$I_p^{\mu} = \mathrm{Tr}[D^{\mu}(X_{i_1})D^{\mu}(X_{i_2}) \ldots D^{\mu}(X_{i_p})]X^{i_1}X^{i_2} \ldots X^{i_p}. \tag{2.12}$$

Clearly $I_p^{\phi} = (-1)^p C_p$ and $I_2^{\phi} = C_2$.

It has been shown by Casimir (1931), Racah (1951) and Gruber and O'Raifeartaigh (1964) that C_2, C_p and I_p^{μ} are all invariant, commuting with all elements of the Lie group under consideration. Thus by Schür's Lemmas, each of these operators is a multiple of the identity within each irreducible representation $\boldsymbol{\lambda}$ of the Lie group. Their corresponding eigenvalues may be denoted by $C_2(\boldsymbol{\lambda})$, $C_p(\boldsymbol{\lambda})$ and $I_p^{\mu}(\boldsymbol{\lambda})$ respectively.

The remarkable result derived by Okubo (1977) takes the form

$$I_p^{\mu}(\boldsymbol{\lambda}) = \sum_{\nu} K_{\lambda\mu}^{\nu} \frac{d(\boldsymbol{\nu})}{d(\boldsymbol{\lambda})} \{\tfrac{1}{2}[C_2(\boldsymbol{\nu}) - C_2(\boldsymbol{\lambda}) - C_2(\boldsymbol{\mu})]\}^p \tag{2.13}$$

where $d(\boldsymbol{\lambda})$ denotes the dimension of the irreducible representation $\boldsymbol{\lambda}$, and the Kronecker product multiplicities $K_{\lambda\mu}^{\nu}$ are defined through the product reduction formula

$$\boldsymbol{\lambda} \times \boldsymbol{\mu} = \sum_{\nu} K_{\lambda\mu}^{\nu} \boldsymbol{\nu}. \tag{2.14}$$

Before discussing the ease with which (2.13) leads both to explicit formulae for $I_p^{\mu}(\boldsymbol{\lambda})$ and to general results concerning the dependence of $I_p^{\mu}(\boldsymbol{\lambda})$ on the components of λ, some trivial special cases should be noted:

$$I_0^{\mu}(\boldsymbol{\lambda}) = d(\boldsymbol{\mu}) \tag{2.15}$$

$$I_p^{\mu}(\mathbf{0}) = 0 \qquad \text{for } p \geqslant 1 \tag{2.16}$$

$$I_p^{0}(\boldsymbol{\lambda}) = 0 \qquad \text{for } p \geqslant 1 \tag{2.17}$$

where $\mathbf{0}$ denotes the trivial identity representation of the group which is such that $D^{0}(X_i) = 0$, $d(\mathbf{0}) = 1$, $C_2(\mathbf{0}) = 0$, $\mathbf{0} \times \boldsymbol{\mu} = \boldsymbol{\mu}$ for all $\boldsymbol{\mu}$ and $\boldsymbol{\lambda} \times \mathbf{0} = \boldsymbol{\lambda}$ for all $\boldsymbol{\lambda}$. For all semi-simple Lie groups $I_1^{\mu}(\boldsymbol{\lambda}) = 0$ since the corresponding representation matrices $D^{\mu}(X_i)$ are traceless. In the case of the full unitary group $\mathrm{U}(N)$, however, the additional generator is essentially the unit matrix for every irreducible representation so that $I_1^{\mu}(\boldsymbol{\lambda}) = d(\boldsymbol{\mu})$.

3. The generalised Popov–Perelemov formula

In order to make use of (2.13) it is necessary to evaluate only $d(\boldsymbol{\lambda})$, $C_2(\boldsymbol{\lambda})$ and $K_{\lambda\mu}^{\nu}$. The dimension of an irreducible representation $\boldsymbol{\lambda}$ is given by the formula, due to Weyl (1926),

$$d(\boldsymbol{\lambda}) = \prod_{r>0} \frac{\boldsymbol{r} \cdot (\boldsymbol{\lambda} + \boldsymbol{\delta})}{\boldsymbol{r} \cdot \boldsymbol{\delta}} \tag{3.1}$$

where the product is taken over the positive roots $\boldsymbol{r}$ of the corresponding Lie algebra, $\boldsymbol{\lambda}$ is the highest weight of the irreducible representation under consideration and

$$\boldsymbol{\delta}=\tfrac{1}{2}\sum_{r>0}\boldsymbol{r}. \tag{3.2}$$

The eigenvalue of the second-order Casimir operator is known through the result due to Racah (1950):

$$C_2(\boldsymbol{\lambda})=(1/N_{\phi})\boldsymbol{\lambda}\,.\,(\boldsymbol{\lambda}+2\boldsymbol{\delta}) \tag{3.3}$$

with the normalisation factor,

$$N_{\phi}=\boldsymbol{\phi}\,.\,(\boldsymbol{\phi}+2\boldsymbol{\delta}), \tag{3.4}$$

fixed by noting that the definition (2.7) together with the nature of the adjoint representation matrices (2.10) imply that $C_2(\boldsymbol{\phi})=1$.

The Kronecker product multiplicities may be found in many ways, but in order to derive general results it is useful to consider the method due to Racah (1964) which depends crucially on the symmetry group W, introduced by Weyl (1926), of the particular Lie algebra in question. This is the symmetry group of the root diagram, whose elements, S, are generated by reflections, S_r, in the hyperplanes perpendicular to the roots $\boldsymbol{r}$:

$$S_r:\boldsymbol{v}\to S_r\boldsymbol{v}=\boldsymbol{v}-2\frac{(\boldsymbol{v}\,.\,\boldsymbol{r})}{(\boldsymbol{r}\,.\,\boldsymbol{r})}\boldsymbol{r}. \tag{3.5}$$

An element S of W is said to have parity $(-1)^{\eta_S}$ which is $+1$ or -1 according as the number of such reflections S_r generating S is even or odd respectively. With this notation, the character formula due to Weyl (1926), from which (3.1) is derived, is

$$\chi^{\lambda}(\boldsymbol{\theta})=\frac{\sum_{S\in\mathrm{W}}(-1)^{\eta_S}\,\mathrm{e}^{\mathrm{i}S(\boldsymbol{\lambda}+\boldsymbol{\delta}).\boldsymbol{\theta}}}{\sum_{S\in\mathrm{W}}(-1)^{\eta_S}\,\mathrm{e}^{\mathrm{i}S\boldsymbol{\delta}.\boldsymbol{\theta}}} \tag{3.6}$$

where $\boldsymbol{\theta}$ is a set of real class parameters. This may be expanded in the form

$$\chi^{\lambda}(\boldsymbol{\theta})=\sum_{\boldsymbol{w}}M_{\boldsymbol{w}}^{\lambda}\,\mathrm{e}^{\mathrm{i}\boldsymbol{w}.\boldsymbol{\theta}} \tag{3.7}$$

where $M_{\boldsymbol{w}}^{\lambda}$ is the multiplicity of the weight $\boldsymbol{w}$ in the representation $\boldsymbol{\lambda}$.

The Weyl symmetry of (3.6) is such that

$$\chi^{S(\boldsymbol{\lambda}+\boldsymbol{\delta})-\boldsymbol{\delta}}(\boldsymbol{\theta})=(-1)^{\eta_S}\chi^{\lambda}(\boldsymbol{\theta}) \tag{3.8}$$

and

$$M_{S\boldsymbol{w}}^{\lambda}=M_{\boldsymbol{w}}^{\lambda} \tag{3.9}$$

for all S in W. It is then straightforward to show, following Racah (1964), that

$$K_{\lambda\mu}^{\nu}=\sum_{S\in\mathrm{W}}(-1)^{\eta_S}M_{\boldsymbol{w}}^{\mu} \tag{3.10}$$

where

$$\boldsymbol{w}=S(\boldsymbol{\nu}+\boldsymbol{\delta})-\boldsymbol{\lambda}-\boldsymbol{\delta}. \tag{3.11}$$

The equivalence relation (3.8) between irreducible representations labelled by $\boldsymbol{\lambda}$ and by $S(\boldsymbol{\lambda}+\boldsymbol{\delta})-\boldsymbol{\delta}$ is such that (3.11) implies

$$d(\boldsymbol{\nu})=(-1)^{\eta_S}d(\boldsymbol{\lambda}+\boldsymbol{w}). \tag{3.12}$$

Moreover (3.5) is simply a reflection for which

$$(S\boldsymbol{u}).(S\boldsymbol{v})=\boldsymbol{u}.\boldsymbol{v}, \tag{3.13}$$

for any vectors $\boldsymbol{u}$ and $\boldsymbol{v}$.

It is then easy to see that (3.3) and (3.11) lead to the result

$$C_2(\boldsymbol{\nu})=C_2(\boldsymbol{\lambda}+\boldsymbol{w}). \tag{3.14}$$

The identities (3.10), (3.12) and (3.14) associated with (3.11) may be incorporated in (2.13). That the resulting summations over $\boldsymbol{\nu}$ and S may be replaced by a single summation over all weights $\boldsymbol{w}$ is implied by Speiser (1964) and relies for its validity on two observations. Firstly each contribution to $K^{\nu}_{\lambda\mu}$ arises from some weight $\boldsymbol{w}$ for which the corresponding transformation S in (3.11) is unique. Secondly, although the summation over all weights $\boldsymbol{w}$ includes additional terms for which no element S exists for any $\boldsymbol{\nu}$ in (3.11), in these cases $d(\boldsymbol{\lambda}+\boldsymbol{w})=0$ so that their inclusion in the final sum makes no difference to its value.

Incorporation of (3.10), (3.12) and (3.14) in (2.13) then yields

$$I^{\mu}_{p}(\boldsymbol{\lambda})=\sum_{\boldsymbol{w}}M^{\mu}_{\boldsymbol{w}}\frac{d(\boldsymbol{\lambda}+\boldsymbol{w})}{d(\boldsymbol{\lambda})}\{\tfrac{1}{2}[C_2(\boldsymbol{\lambda}+\boldsymbol{w})-C_2(\boldsymbol{\lambda})-C_2(\boldsymbol{\mu})]\}^{p}. \tag{3.15}$$

Some cancellations between the numerator and denominator may be effected by noting that

$$\frac{d(\boldsymbol{\lambda}+\boldsymbol{w})}{d(\boldsymbol{\lambda})}=\prod_{\substack{\boldsymbol{r}>0\\ \boldsymbol{r}.\boldsymbol{w}\neq 0}}\frac{\boldsymbol{r}.(\boldsymbol{\lambda}+\boldsymbol{w}+\boldsymbol{\delta})}{\boldsymbol{r}.(\boldsymbol{\lambda}+\boldsymbol{\delta})}. \tag{3.16}$$

Furthermore it is known (Racah 1964) that if $\boldsymbol{w}$ is a weight of the representation $\boldsymbol{\mu}$ and $\boldsymbol{r}.\boldsymbol{w}\neq 0$ then $\boldsymbol{v}$ is also a weight of $\boldsymbol{\mu}$ where

$$\boldsymbol{v}=S_r\boldsymbol{w}=\boldsymbol{w}-2\left(\frac{\boldsymbol{w}.\boldsymbol{r}}{\boldsymbol{r}.\boldsymbol{r}}\right)\boldsymbol{r}. \tag{3.17}$$

Eliminating $\boldsymbol{r}$ in favour of $\boldsymbol{v}$ gives

$$\frac{d(\boldsymbol{\lambda}+\boldsymbol{w})}{d(\boldsymbol{\lambda})}=\prod_{\boldsymbol{v}=S_r\boldsymbol{w}}\frac{(\boldsymbol{w}-\boldsymbol{v}).(\boldsymbol{\lambda}+\boldsymbol{\delta})+\frac{1}{2}(\boldsymbol{w}-\boldsymbol{v}).(\boldsymbol{w}-\boldsymbol{v})}{(\boldsymbol{w}-\boldsymbol{v}).(\boldsymbol{\lambda}+\boldsymbol{\delta})} \tag{3.18}$$

where the product is taken over all those weights $\boldsymbol{v}$ obtained from $\boldsymbol{w}$ by means of a single Weyl reflection. Making use of this result and (3.3) in (3.15) then gives

$$I^{\mu}_{p}(\boldsymbol{\lambda})=\sum_{\boldsymbol{w}}M^{\mu}_{\boldsymbol{w}}\left[-\frac{p_{\boldsymbol{w}}(\boldsymbol{\lambda})}{N_{\phi}}\right]^{p}\prod_{\boldsymbol{v}=S_r\boldsymbol{w}}\frac{p_{\boldsymbol{v}}(\boldsymbol{\lambda})-p_{\boldsymbol{w}}(\boldsymbol{\lambda})+\sigma_{\boldsymbol{vw}}}{p_{\boldsymbol{v}}(\boldsymbol{\lambda})-p_{\boldsymbol{w}}(\boldsymbol{\lambda})} \tag{3.19}$$

where

$$p_{\boldsymbol{w}}(\boldsymbol{\lambda})=-\tfrac{1}{2}N_{\phi}[C_2(\boldsymbol{\lambda}+\boldsymbol{w})-C_2(\boldsymbol{\lambda})-C_2(\boldsymbol{\mu})]=-\boldsymbol{w}.(\boldsymbol{\lambda}+\boldsymbol{\delta})+\tfrac{1}{2}(\boldsymbol{\mu}.\boldsymbol{\mu}+2\boldsymbol{\mu}.\boldsymbol{\delta}-\boldsymbol{w}.\boldsymbol{w}) \tag{3.20}$$

so that

$$p_{\boldsymbol{v}}(\boldsymbol{\lambda})-p_{\boldsymbol{w}}(\boldsymbol{\lambda})=(\boldsymbol{w}-\boldsymbol{v}).(\boldsymbol{\lambda}+\boldsymbol{\delta}) \tag{3.21}$$

whilst

$$\sigma_{vw} = \tfrac{1}{2}(\boldsymbol{w}-\boldsymbol{v}).(\boldsymbol{w}-\boldsymbol{v}). \tag{3.22}$$

This formula (3.19) represents the generalisation to any semi-simple Lie group of the results obtained previously for the classical groups in the case $\boldsymbol{\mu}=\boldsymbol{\omega}$ or $\bar{\boldsymbol{\omega}}$, the contragradient of the defining representation, by Popov and Perelemov (1967), Louck and Biedenharn (1970), Bracken and Green (1971), Okubo (1975), Nwachuku and Rashid (1976) and rederived using the present methods by Okubo (1977) and Edwards (1978). It should be noted that the result is a sum of rational terms, each of which involves a denominator consisting of a number of factors. The total number of factors in each term is given by the number of roots $\boldsymbol{r}$ which are not perpendicular to a particular fixed weight $\boldsymbol{w}$.

4. Weyl symmetry properties

Whilst the result of the previous section is extremely powerful, its use depends upon a knowledge of both the weight multiplicities M^{μ}_{w} and the roots $\boldsymbol{r}$. Their introduction serves to eliminate the need for determining the Kronecker product multiplicities $K^{\nu}_{\lambda\mu}$ by other methods. Although this is helpful the final result has two defects. Firstly the symmetry of the result under the action of the Weyl group elements is not made manifest and secondly the formula is still, even with some cancellations made, rational rather than polynomial in the components of $\boldsymbol{\lambda}$. That this can be remedied may be shown by expressing all the results in terms of the vector

$$\boldsymbol{l}=\boldsymbol{\lambda}+\boldsymbol{\delta} \tag{4.1}$$

on which the Weyl group elements S act, as in the character formula (3.6), and through which the dependence upon $\boldsymbol{\lambda}$ can be implicitly expressed.

To this end all the quantities $d(\boldsymbol{\lambda})$, $C_2(\boldsymbol{\lambda})$, $I^{\mu}_{p}(\boldsymbol{\lambda})$, $p_v(\boldsymbol{\lambda})$ and $\chi^{\lambda}(\boldsymbol{\theta})$ will be denoted throughout this section by $d(\boldsymbol{l})$, $C_2(\boldsymbol{l})$, $I^{\mu}_{p}(\boldsymbol{l})$, $p_v(\boldsymbol{l})$ and $\chi^{l}(\boldsymbol{\theta})$ respectively. No confusion should arise and in this form it is easy to express the Weyl group action.

Thus, for example,

$$\begin{aligned} C_2(\boldsymbol{l}) &\equiv C_2(\boldsymbol{\lambda}) = (1/N_\phi)\boldsymbol{\lambda}.(\boldsymbol{\lambda}+2\boldsymbol{\delta}) \\ &= (1/N_\phi)[(\boldsymbol{\lambda}+\boldsymbol{\delta}).(\boldsymbol{\lambda}+\boldsymbol{\delta})-\boldsymbol{\delta}.\boldsymbol{\delta}] = (1/N_\phi)(\boldsymbol{l}.\boldsymbol{l}-\boldsymbol{\delta}.\boldsymbol{\delta}), \end{aligned} \tag{4.2}$$

so that directly from (3.13)

$$C_2(S\boldsymbol{l}) = C_2(\boldsymbol{l}). \tag{4.3}$$

Similarly (3.8) implies

$$\chi^{Sl}(\boldsymbol{\theta}) = (-1)^{n_S}\chi^{l}(\boldsymbol{\theta}), \tag{4.4}$$

and hence

$$d(S\boldsymbol{l}) = (-1)^{n_S} d(\boldsymbol{l}). \tag{4.5}$$

In order to make manifest the Weyl symmetry of $I^{\mu}_{p}(\boldsymbol{\lambda})$ it is necessary to return to (3.15) and to consider the subsets of weights of $\boldsymbol{\mu}$, each consisting of a set of weights $\{\boldsymbol{w}: S\boldsymbol{w}=\boldsymbol{\kappa}$ with $S\in \mathrm{W}\}$ such that $\boldsymbol{\kappa}$ is the highest weight of the set, i.e. a dominant weight. The possible values of $\boldsymbol{\kappa}$ are precisely those of the irreducible representation labels: hence the use of a Greek letter. By virtue of the Weyl symmetry, if $\boldsymbol{w}$ is a weight in such a set,

$$M_{\boldsymbol{w}}^{\boldsymbol{\mu}} = M_{S^{-1}\boldsymbol{\kappa}}^{\boldsymbol{\mu}} = M_{\boldsymbol{\kappa}}^{\boldsymbol{\mu}} \qquad \text{for all } S \in \mathrm{W}. \tag{4.6}$$

Furthermore in (3.15)

$$d(\boldsymbol{\lambda}+\boldsymbol{w}) \equiv d(\boldsymbol{l}+\boldsymbol{w}) = d(\boldsymbol{l}+S^{-1}\boldsymbol{\kappa}) = (-1)^{\eta_S} d(S\boldsymbol{l}+\boldsymbol{\kappa}) \tag{4.7}$$

and

$$d(\boldsymbol{\lambda}) \equiv d(\boldsymbol{l}) = (-1)^{\eta_S} d(S\boldsymbol{l}). \tag{4.8}$$

Finally

$$\begin{aligned} p_{\boldsymbol{w}}(\boldsymbol{\lambda}) \equiv p_{\boldsymbol{w}}(\boldsymbol{l}) = p_{S^{-1}\boldsymbol{\kappa}}(\boldsymbol{l}) &= -\boldsymbol{w}\,.\,\boldsymbol{l} + \tfrac{1}{2}(\boldsymbol{\mu}\,.\,\boldsymbol{\mu} + 2\boldsymbol{\mu}\,.\,\boldsymbol{\delta} - \boldsymbol{w}\,.\,\boldsymbol{w}) \\ &= -(S\boldsymbol{w})\,.\,(S\boldsymbol{l}) + \tfrac{1}{2}(\boldsymbol{\mu}\,.\,\boldsymbol{\mu} + 2\boldsymbol{\mu}\,.\,\boldsymbol{\delta} - \boldsymbol{w}\,.\,\boldsymbol{w}) \\ &= -\boldsymbol{\kappa}\,.\,(S\boldsymbol{l}) + \tfrac{1}{2}(\boldsymbol{\mu}\,.\,\boldsymbol{\mu} + 2\boldsymbol{\mu}\,.\,\boldsymbol{\delta} - \boldsymbol{\kappa}\,.\,\boldsymbol{\kappa}) = p_{\boldsymbol{\kappa}}(S\boldsymbol{l}). \end{aligned} \tag{4.9}$$

Hence

$$I_p^{\boldsymbol{\mu}}(\boldsymbol{\lambda}) \equiv I_p^{\boldsymbol{\mu}}(\boldsymbol{l}) = \sum_{\boldsymbol{\kappa}} M_{\boldsymbol{\kappa}}^{\boldsymbol{\mu}} \frac{1}{|\mathrm{W}_{\boldsymbol{\kappa}}|} \sum_{S\in\mathrm{W}} \left(\frac{-p_{\boldsymbol{\kappa}}(S\boldsymbol{l})}{N_\phi}\right)^p \frac{d(S\boldsymbol{l}+\boldsymbol{\kappa})}{d(S\boldsymbol{l})} \tag{4.10}$$

where $|\mathrm{W}_{\boldsymbol{\kappa}}| = |\{S : S\boldsymbol{\kappa} = \boldsymbol{\kappa}, S \in \mathrm{W}\}|$ is the number of elements of the Weyl group leaving $\boldsymbol{\kappa}$ invariant.

This formula manifests the Weyl symmetry of the generalised Casimir operator eigenvalues:

$$I_p^{\boldsymbol{\mu}}(S\boldsymbol{l}) = I_p^{\boldsymbol{\mu}}(\boldsymbol{l}). \tag{4.11}$$

Moreover the dependence upon $\boldsymbol{l}$ arises from terms of the form

$$\frac{1}{d(\boldsymbol{l})} \sum_{S\in\mathrm{W}} (-1)^{\eta_S} (\boldsymbol{\kappa}\,.\,S\boldsymbol{l})^q\, d(S\boldsymbol{l}+\boldsymbol{\kappa}) \tag{4.12}$$

where use has been made of (4.5) and (4.9) together with a binomial expansion of $[p_{\boldsymbol{\kappa}}(S\boldsymbol{l})]^p$. By virtue of (3.1) the denominator is factorised into terms of the form $(\boldsymbol{r}\,.\,\boldsymbol{l})$ for each root $\boldsymbol{r}$. That each such factor appears in the numerator may be seen by noting that

$$\sum_{S\in\mathrm{W}} (-1)^{\eta_S} [\boldsymbol{\kappa}\,.\,(S\boldsymbol{l})]^q d(S\boldsymbol{l}+\boldsymbol{\kappa}) = \sum_{S\in\mathrm{W}} [(S\boldsymbol{\kappa})\,.\,\boldsymbol{l}]^q d(\boldsymbol{l}+S\boldsymbol{\kappa})$$

where the action of S on $\boldsymbol{l}$ has been replaced by the action of S^{-1} on $\boldsymbol{\kappa}$, with the summation variable subsequently changed from S^{-1} back to S for convenience. This expression may then be written in the form

$$\sum_{T\in\mathrm{W}/\{I,S_r\}} \{[(T\boldsymbol{\kappa})\,.\,\boldsymbol{l}]^q\, d(\boldsymbol{l}+T\boldsymbol{\kappa}) + [(S_rT\boldsymbol{\kappa})\,.\,\boldsymbol{l}]^q\, d(\boldsymbol{l}+S_rT\boldsymbol{\kappa})\}$$

by making use of the coset decomposition of W with respect to the subgroup $\{I, S_r\}$ of order two associated with any positive root $\boldsymbol{r}$. The elements, T, are the corresponding coset representatives. Using (3.13) and (4.5) then yields

$$\sum_{T\in\mathrm{W}/\{I,S_r\}} \{[(T\boldsymbol{\kappa})\,.\,\boldsymbol{l}]^q\, d(\boldsymbol{l}+T\boldsymbol{\kappa}) - [(T\boldsymbol{\kappa})\,.\,S_r\boldsymbol{l}]^q\, d(S_r\boldsymbol{l}+T\boldsymbol{\kappa})\}. \tag{4.13}$$

This expression vanishes identically if $(\boldsymbol{l}\,.\,\boldsymbol{r})=0$ since in such a case

$$S_r\boldsymbol{l}=\boldsymbol{l}-2\left(\frac{\boldsymbol{l}\,.\,\boldsymbol{r}}{\boldsymbol{r}\,.\,\boldsymbol{r}}\right)\boldsymbol{r}=\boldsymbol{l}. \tag{4.14}$$

Thus the polynomial structure guarantees, as required, that this numerator contains $(\boldsymbol{r}\,.\,\boldsymbol{l})$ as a factor. Therefore each term (4.12) appearing in the expansion of (4.10) must be simply a polynomial in the components of $\boldsymbol{l}$ which from (4.11) is seen to be symmetric under the action of the Weyl group on $\boldsymbol{l}$.

This structure is illustrated in the case of the group G_2 in the next section and is used to simplify the calculation of all generalised Casimir operator eigenvalues in § 6. Before proceeding to this, two special terms in (4.10) will be discussed. Firstly if $\boldsymbol{\kappa}=\boldsymbol{0}$ then clearly $|W_{\boldsymbol{\kappa}}|=|W_{\boldsymbol{0}}|=|W|$ whilst $p_{\boldsymbol{\kappa}}(S\boldsymbol{l})=p_{\boldsymbol{0}}(S\boldsymbol{l})=\frac{1}{2}\boldsymbol{\mu}\,.\,(\boldsymbol{\mu}+2\boldsymbol{\delta})=\frac{1}{2}N_{\boldsymbol{\phi}}C_2(\boldsymbol{\mu})$. Secondly if $\boldsymbol{\kappa}=\boldsymbol{\mu}$ then $p_{\boldsymbol{\kappa}}(S\boldsymbol{l})=p_{\boldsymbol{\mu}}(S\boldsymbol{l})=\boldsymbol{\mu}\,.\,(-S\boldsymbol{l}+\boldsymbol{\delta})$. These two results make it very easy to cope with all invariants $I_p^{\boldsymbol{\mu}}(\boldsymbol{\lambda})$ in cases for which $\boldsymbol{\mu}$ contains only one non-zero dominant weight $\boldsymbol{\kappa}=\boldsymbol{\mu}$. This is true for the defining representation $\boldsymbol{\omega}$ and its contragradient $\bar{\boldsymbol{\omega}}$ of any semi-simple Lie group or of $U(N)$, for the adjoint representations $\boldsymbol{\phi}$ of $U(k)$, $SU(k+1)$, $SO(2k)$, E_6, E_7 and E_8, for the fundamental representations $\{1^m\}$ of $SU(k+1)$ and $U(k+1)$ with $m=1,2,\ldots,k$, for the representation $\langle 1^2\rangle$ of $Sp(2k)$, and for the spin representations Δ of $SO(2k+1)$ and $\Delta_\pm$ of $SO(2k)$.

5. Application to G_2

In the case of the exceptional Lie group G_2 it is simplest to choose $\boldsymbol{\mu}$ to be the defining seven-dimensional representation $\boldsymbol{\omega}=[1,0]$, where the notation of King and Qubanchi (1978) has been adopted, which conforms with that of Wybourne (1970, p 46). The required Kronecker product is

$$\begin{aligned}[\rho_1,\rho_2]\times[1,0]=&[\rho_1,\rho_2]+[\rho_1+1,\rho_2]+[\rho_1-1,\rho_2+1]\\&+[\rho_1,\rho_2-1]+[\rho_1-1,\rho_2]+[\rho_1+1,\rho_2-1]+[\rho_1,\rho_2+1],\end{aligned} \tag{5.1}$$

subject where necessary to the modification rules or equivalence relations

$$[\nu_1,\nu_2]=-[\nu_2-1,\nu_1+1]=-[\nu_1+\nu_2+1,-\nu_2-2]. \tag{5.2}$$

The dimension formula and the expression for the eigenvalues of C_2 are

$$d[\nu_1,\nu_2]=(1/5!)(2\nu_1+\nu_2+5)(\nu_1+2\nu_2+4)(\nu_1+\nu_2+3)(\nu_1-\nu_2+1)(\nu_1+2)(\nu_2+1) \tag{5.3}$$

and

$$C_2[\nu_1,\nu_2]=\tfrac{1}{12}[\nu_1(\nu_1+5)+\nu_2(\nu_2+4)+\nu_1\nu_2]. \tag{5.4}$$

These both remain valid under the modifications corresponding to (5.2), which are themselves determined by the action of the Weyl group.

Hence from the formula (2.13), due to Okubo (1977),

$$\begin{aligned}I_p^{[1,0]}[\rho_1,\rho_2]=&\{d[\rho_1,\rho_2](-6)^p+d[\rho_1+1,\rho_2](2\rho_1+\rho_2)^p\\&+d[\rho_1-1,\rho_2+1](-\rho_1+\rho_2-6)^p+d[\rho_1,\rho_2-1](-\rho_1-2\rho_2-9)^p\\&+d[\rho_1-1,\rho_2](-2\rho_1-\rho_2-10)^p+d[\rho_1+1,\rho_2-1](\rho_1-\rho_2-4)^p\\&+d[\rho_1,\rho_2+1](\rho_1+2\rho_2-1)^p\}/d[\rho_1,\rho_2]24^p.\end{aligned} \tag{5.5}$$

Whilst this formula is very easy to use in conjunction with (5.3) it does not make manifest the Weyl symmetry. This is a result of the adoption of the labels $[\rho_1, \rho_2]$ to specify the irreducible representations of G_2. It is more convenient, as stressed in the previous section, to work in terms of the vector $\boldsymbol{l}$. This requires first the identification of the root space. Although G_2 is of rank two it is helpful to embed the root space in a three-dimensional Euclidean space. In terms of mutually orthogonal unit vectors $\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3$, the positive root vectors are $\boldsymbol{e}_1-\boldsymbol{e}_2$, $\boldsymbol{e}_1-\boldsymbol{e}_3$, $\boldsymbol{e}_2-\boldsymbol{e}_3$, $\frac{1}{3}(2\boldsymbol{e}_1-\boldsymbol{e}_2-\boldsymbol{e}_3)$, $\frac{1}{3}(\boldsymbol{e}_1-2\boldsymbol{e}_2+\boldsymbol{e}_3)$ and $\frac{1}{3}(\boldsymbol{e}_1+\boldsymbol{e}_2-2\boldsymbol{e}_3)$, so that $\boldsymbol{\delta}=\frac{1}{3}(5\boldsymbol{e}_1-\boldsymbol{e}_2-4\boldsymbol{e}_3)$. The highest weight labelling scheme for irreducible representations is then such that $\boldsymbol{\lambda}=(\lambda_1, \lambda_2, \lambda_3)$ with $\lambda_1+\lambda_2+\lambda_3=0$, whilst both $\lambda_2-\lambda_3$ and $\frac{1}{3}(\lambda_1-2\lambda_2+\lambda_3)$ are non-negative integers. The relationship between this notation and the labels $[\rho_1, \rho_2]$ used earlier and the labels (m_1, m_2) of Okubo (1977) is such that

$$\begin{aligned}
\lambda_1 &= \tfrac{1}{3}(2\rho_1+\rho_2)=\tfrac{1}{3}(3m_1+2m_2)\\
\lambda_2 &= \tfrac{1}{3}(-\rho_1+\rho_2)=\tfrac{1}{3}(-m_2)\\
\lambda_3 &= \tfrac{1}{3}(-\rho_1-2\rho_2)=\tfrac{1}{3}(-3m_1-m_2)
\end{aligned} \tag{5.6}$$

so that $\rho_1=\lambda_1-\lambda_2$, $\rho_2=\lambda_2-\lambda_3$, $m_1=\lambda_1+2\lambda_2$, and $m_2=-3\lambda_2$. Finally $\boldsymbol{l}$ is introduced in the usual way via (4.1) so that $l_1=\lambda_1+\frac{5}{3}$, $l_2=\lambda_2-\frac{1}{3}$ and $l_3=\lambda_3-\frac{4}{3}$.

With this notation,

$$d(\boldsymbol{l})=\tfrac{9}{40}l_1l_2l_3(l_1-l_2)(l_1-l_3)(l_2-l_3)=\tfrac{9}{40}(l_1^2-l_2^2)(l_2^2-l_3^2)(l_3^2-l_1^2) \tag{5.7}$$

and

$$C_2(\boldsymbol{l})=\tfrac{1}{8}(l_1^2+l_2^2+l_3^2-\tfrac{14}{3}), \tag{5.8}$$

illustrating the symmetry associated with the Weyl operations on $\boldsymbol{l}$ which permute the components and change the sign of $\boldsymbol{l}$, as discussed by King and Qubanchi (1978).

The formula (3.15) then yields

$$\begin{aligned}
I_p^{\omega}(\boldsymbol{l}) = [&d(l_1, l_2, l_3)(-6)^p+d(l_1+\tfrac{2}{3}, l_2-\tfrac{1}{3}, l_3-\tfrac{1}{3})(3l_1-5)^p\\
&\qquad+d(l_1-\tfrac{1}{3}, l_2+\tfrac{2}{3}, l_3-\tfrac{1}{3})(3l_2-5)^p\\
&+d(l_1-\tfrac{1}{3}, l_2-\tfrac{1}{3}, l_3+\tfrac{2}{3})(3l_3-5)^p+d(l_1-\tfrac{2}{3}, l_2+\tfrac{1}{3}, l_3+\tfrac{1}{3})(-3l_1-5)^p\\
&+d(l_1+\tfrac{1}{3}, l_2-\tfrac{2}{3}, l_3+\tfrac{1}{3})(-3l_2-5)^p\\
&+d(l_1+\tfrac{1}{3}, l_2+\tfrac{1}{3}, l_3-\tfrac{2}{3})(-3l_3-5)^p]/d(l_1, l_2, l_3)\,.\,24^p,
\end{aligned} \tag{5.9}$$

whilst (3.19) takes the form

$$I_p^{\omega}(\boldsymbol{l})=\left(-\frac{1}{24}\right)^p\sum_{i=1}^{7}p_i^p\prod_{\substack{j=1\\ j\neq i}}^{7}\frac{(p_j-p_i+\sigma_{ji})}{(p_j-p_i)} \tag{5.10}$$

with

$$p_i=\begin{cases}6 & \text{for } i=1\\ -3l_{i-1}+5 & \text{for } i=2,3,4\\ 3l_{i-4}+5 & \text{for } i=5,6,7\end{cases}$$

and

$$\sigma = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 3 & 4 & 1 & 1 \\ 0 & 3 & 0 & 3 & 1 & 4 & 1 \\ 0 & 3 & 3 & 0 & 1 & 1 & 4 \\ 0 & 4 & 1 & 1 & 0 & 3 & 3 \\ 0 & 1 & 4 & 1 & 3 & 0 & 3 \\ 0 & 1 & 1 & 4 & 3 & 3 & 0 \end{pmatrix}.$$

This formula is the direct analogue of the formula given by Okubo (1977) for each of the classical Lie groups. It is the solution to the problem imposed implicitly by Popov and Perelemov (1968) who, for the group G_2, gave the eigenvalues of $I_p^{\omega}(\boldsymbol{l})$ as the sum of all the elements of the pth power of a square matrix having p_i as its diagonal elements for $i = 1, 2, \ldots, 7$.

In order to obtain the Weyl-symmetric polynomial expansion of $I_p^{\omega}(\boldsymbol{l})$ it is most convenient to use (4.10). In the case $\boldsymbol{\mu} = \boldsymbol{\omega} = (\frac{2}{3}, -\frac{1}{3}, -\frac{1}{3})$ all the multiplicities are 1 and the sum is over just two values of $\boldsymbol{\kappa}$: namely $\boldsymbol{\kappa} = (\frac{2}{3}, -\frac{1}{3}, -\frac{1}{3})$ and $\boldsymbol{\kappa} = (0, 0, 0)$. This gives

$$I_p^{\omega}(\boldsymbol{l}) = (-\tfrac{1}{4})^p + \frac{1}{2d(\boldsymbol{l})} \sum_{S \in W} \{8^{-p}(Sl_1 - \tfrac{5}{3})^p (-1)^{\eta_S} d(Sl_1 + \tfrac{2}{3}, Sl_2 - \tfrac{1}{3}, Sl_3 - \tfrac{1}{3})\} \tag{5.11}$$

where $Sl_i = (S\boldsymbol{l})_i$ and the Weyl symmetry operations may be exhibited in the form

$$\begin{aligned} \sum_{S \in W} (-1)^{\eta_S} \boldsymbol{l} = {} & (l_1, l_2, l_3) + (l_2, l_3, l_1) + (l_3, l_1, l_2) - (l_2, l_1, l_3) - (l_3, l_2, l_1) - (l_1, l_3, l_2) \\ & + (-l_1, -l_2, -l_3) + (-l_2, -l_3, -l_1) + (-l_3, -l_1, -l_2) - (-l_2, -l_1, -l_3) \\ & - (-l_3, -l_2, -l_1) - (-l_1, -l_3, -l_2). \end{aligned} \tag{5.12}$$

Hence the sum in (5.11) gives an antisymmetric function of $\boldsymbol{l}$, invariant under reflection. From the previous section, it contains a factor $d(\boldsymbol{l})$. Since, from the explicit form (5.7), this factor is also symmetric and invariant under reflection, so is the quotient. Moreover these remarks apply to each term in a binomial expansion of $(Sl_1 - \frac{5}{3})^p$ in powers of Sl_1, which is convenient for the explicit evaluation of the sum. Thus

$$\sum_{S \in W} (-1)^{\eta_S} (Sl_1)^q \, d(Sl_1 + \tfrac{2}{3}, Sl_2 - \tfrac{1}{3}, Sl_3 - \tfrac{1}{3}) = A_q(\boldsymbol{l}) \, d(\boldsymbol{l}) \tag{5.13}$$

where $A_q(\boldsymbol{l}) = A_q(-\boldsymbol{l})$ is a symmetric polynomial in l_1, l_2 and l_3 of degree at most q. In the relevant three-dimensional space $A_q(\boldsymbol{l})$ can be expressed in terms of the power sum symmetric functions

$$\sigma_m = l_1^m + l_2^m + l_3^m \qquad \text{with } m = 0, 1, 2, 3. \tag{5.14}$$

For these functions the constraint $l_1 + l_2 + l_3 = 0$ is the condition $\sigma_1 = 0$ and implies further that $\sigma_4 = \frac{1}{2}\sigma_2^2$ and $\sigma_6 = \frac{1}{3}\sigma_3^2 + \frac{1}{4}\sigma_2^3$. Since $A_q(\boldsymbol{l}) = A_q(-\boldsymbol{l})$, only even powers of σ_3 can appear, so the appropriate complete set of symmetric functions is formed from σ_0, σ_2 and either σ_3^2 or σ_6.

It is found that

$$A_0(\boldsymbol{l})=6 \qquad A_1(\boldsymbol{l})=12 \qquad A_2(\boldsymbol{l})=2(\sigma_2+5)$$
$$A_3(\boldsymbol{l})=(54\sigma_2+40)/9 \qquad A_4(\boldsymbol{l})=(9\sigma_2^2+60\sigma_2+10)/9$$
$$A_5(\boldsymbol{l})=(90\sigma_2^2+100\sigma_2+4)/27 \qquad A_6(\boldsymbol{l})=(36\sigma_6+75\sigma_2^2+20\sigma_2)/18. \tag{5.15}$$

Using these results in (5.11), after the binomial expansion of $(Sl_1-\frac{5}{3})^p$ has been effected, gives with the notation $I_p=I_p^{\omega}(\boldsymbol{l})$

$$I_0=7 \qquad I_1=0 \qquad I_2=(3\sigma_2-14)/96$$
$$I_3=-I_2/4 \qquad I_4=(24I_2^2+7I_2)/96 \qquad I_5=(-360I_2^2-55I_2)/2304 \tag{5.16}$$
$$I_6=(972\sigma_6+6075\sigma_2^2-39\,960\sigma_2+27\,884)/127\,401\,984.$$

Proceeding in the same way with $\boldsymbol{\mu}=\boldsymbol{\phi}=[11]$, the 14-dimensional adjoint representation of G_2 yields for $C_p=(-1)^pI_p^{\phi}(\boldsymbol{l})$

$$C_0=14 \qquad C_1=0 \qquad C_2=(3\sigma_2-14)/24$$
$$C_3=C_2/4 \qquad C_4=(30C_2^2+11C_2)/192 \qquad C_5=(135C_2^2+5C_2)/1152 \tag{5.17}$$
$$C_6=(-37\,908\sigma_6+21\,870\sigma_2^3+47\,385\sigma_2^2-1406\,160\sigma_2+4333\,044)/191\,102\,976,$$

whilst in the case $\boldsymbol{\mu}=[20]$, the 27-dimensional representation, results for $I'_m=I_m^{[20]}(\boldsymbol{l})$ are

$$I'_0=27 \qquad I'_1=0 \qquad I'_2=3(3\sigma_2-14)/32$$
$$I'_3=-I'_2/4 \qquad I'_4=(8I_2'^2+5I'_2)/96 \qquad I'_5=-(440I_2'^2-15I'_2)/6912 \tag{5.18}$$
$$I'_6=(12\,636\sigma_6+4860\sigma_2^3+6075\sigma_2^2-780\,840\sigma_2+2675\,852)/42\,467\,328.$$

These results immediately show that the two algebraically independent Casimir operators of G_2 are of the second and sixth orders, as stated by Racah (1950) and confirmed by Scheibling and Umezawa (1970) and Okubo (1977). The result (5.16) also confirms the conclusion of Scheibling and Umezawa (1970) that I_6 depends essentially upon the symmetric function σ_6, whilst the coefficients given here provide the precise nature of this dependence.

That higher-order invariants give rise to no further algebraically independent invariants may also be tested. For example the list (5.16) may be extended to give

$$I_7=(-1202\,688I_6+12\,960I_2^3+57\,780I_2^2+7041I_2)/995\,328$$

and

$$I_8=(14\,681\,088I_6+10\,616\,832I_6I_2-414\,720I_2^4$$
$$-1054\,080I_2^3-975\,720I_2^2-105\,007I_2)/15\,925\,248,$$

indicating the expected dependence upon I_2 and I_6.

The argument given after equation (5.13), showing that I_p^{ω} is a function of σ_2 and σ_6, applies equally well to I_p^{μ} for any $\boldsymbol{\mu}$. Thus there are only two independent invariants. The factoring of $d(\boldsymbol{l})$ may also be deduced without appealing to the previous section. Application of (5.12) always gives a function which is antisymmetric and invariant under reflection, so for any multinomial $f(\boldsymbol{l})$,

$$\sum_{S\in W}(-1)^{\eta_S}f(\boldsymbol{l})=(l_1-l_2)(l_2-l_3)(l_3-l_1)g(\boldsymbol{l})$$

where g is symmetric and changes sign on reflection. When g is expressed in terms of the elementary symmetric functions, $a_1 = 0$ can be eliminated and then the reflection property requires it to be odd in a_3. Hence g has the factor $a_3(\boldsymbol{l}) = l_1 l_2 l_3 = -(l_2+l_3)(l_3+l_1)(l_1+l_2)$, and $d(\boldsymbol{l})$ is a factor of $\Sigma_{S\in W}(-1)^{\eta_S} f(\boldsymbol{l})$.

6. Expansions in terms of symmetric functions

The work involved in expanding $A_q(\boldsymbol{l})$ for the group G_2 may be avoided if use is made of the completeness of the power sum symmetric functions, the symmetry (4.11) and the polynomial nature of $I_p^{\mu}(\boldsymbol{l})$ subsequently proved in § 4. These facts imply that

$$\alpha_p I_p^{\mu}(\boldsymbol{l}) = \sum_{k=1}^{\eta_p} \beta_p(k) L_k(\boldsymbol{l}) \tag{6.1}$$

where α_p is a normalisation constant, $\beta_p(k)$ is the required expansion coefficient, $L_k(\boldsymbol{l})$ is a polynomial in the components of $\boldsymbol{l}$, and k is an index labelling a complete set of such linearly independent polynomials. The number of these, η_p, depends not only upon the group in question but also upon the order p of the invariant.

Given the required complete set $L_k(\boldsymbol{l})$, the corresponding value of η_p, and η_p independent Kronecker products of the irreducible representation $\boldsymbol{\mu}$ with various representations $\boldsymbol{\lambda}$, it is a straightforward task to evaluate both $I_p^{\mu}(\boldsymbol{l})$ and $L_k(\boldsymbol{l})$ and hence obtain linear equations for the coefficients $\beta_p(k)$. This procedure circumvents the generating function methods used by Popov (1976) and Nwachuku (1979) for the classical groups, and applies equally well to all semi-simple groups.

The functions $L_k(\boldsymbol{l})$ depend upon the group in question and must be invariant under the action of the corresponding Weyl group. The Weyl groups of SU(N), SO($2N+1$), Sp($2N$) and SO($2N$) all contain the symmetric group Σ_N as a subgroup, whilst the Weyl groups of G_2, F_4, E_6, E_7 and E_8 contain the symmetric groups Σ_3, Σ_4, $\Sigma_2\times\Sigma_6$, Σ_8 and Σ_8 respectively as subgroups. Thus, save in the case of E_6, it is natural to use as basis functions symmetric polynomials such as the σ_m used in the case of G_2 in the previous section. However the validity of (2.16) suggests that rather than using σ_m it is preferable to use

$$S_m(\boldsymbol{l}) = \sum_{i=1}^{d} (l_i^m - \delta_i^m), \tag{6.2}$$

so that if $\boldsymbol{\lambda} = \boldsymbol{0}$ then $S_m(\boldsymbol{l}) = 0$. The parameter d is the dimension of the Euclidean space in which the root and weight spaces are embedded. In the case of SU(N), SO($2N+1$), Sp($2N$) and SO($2N$) $d = N$, whilst for G_2 $d = 3$, for F_4 $d = 4$ and for E_6, E_7 and E_8 $d = 8$. In general in such a d-dimensional space the only independent functions $S_m(\boldsymbol{l})$ are those with $m = 1, 2, 3, \ldots, d$. However in the case of G_2 with $d = 3$, as demonstrated in § 5, it is convenient to use only $m = 2$ and $m = 6$. In the case of SO($2N+1$), Sp($2N$), SO($2N$), F_4 and E_8, since the Weyl group includes the inversion $S\boldsymbol{l} = -\boldsymbol{l}$, it is only necessary to use $S_m(\boldsymbol{l})$ with m even, except that for both SO($2N$) and E_8 completeness is only achieved through the inclusion of the elementary symmetric function

$$a_d(\boldsymbol{l}) = \prod_{i=1}^{d} l_i \tag{6.3}$$

Table 2. Irreducible representation labels.

Group	d	$\boldsymbol{\rho}$	$\boldsymbol{\lambda}$	$\boldsymbol{\delta}$	N_Φ
SU($k+1$)	$k+1$	$\{\rho_1, \rho_2, \ldots, \rho_k\}$	$\lambda_i=\rho_i-\bar{\rho}$ for $i=1, 2, \ldots, k$ $\lambda_{k+1}=-\bar{\rho}$ with $\bar{\rho}=(\rho_1+\rho_2+\ldots+\rho_k)/(k+1)$	$\frac{1}{2}(k, k-2, k-4, \ldots, -k)$	$2(k+1)$
SO($2k+1$)	k	$[\rho_1, \rho_2, \ldots, \rho_k]$	$\lambda_i=\rho_i$ for $i=1, 2, \ldots, k$	$\frac{1}{2}(2k-1, 2k-3, \ldots, 3, 1)$	$2(2k-1)$
Sp($2k$)	k	$\langle\rho_1, \rho_2, \ldots, \rho_k\rangle$	$\lambda_1=\rho_i$ for $i=1, 2, \ldots, k$	$(k, k-1, \ldots, 2, 1)$	$2(2k+2)$
SO($2k$)	k	$[\rho_1, \rho_2, \ldots, \rho_k]$	$\lambda_i=\rho_i$ for $i=1, 2, \ldots, k$	$(k-1, k-2, \ldots, 1, 0)$	$2(2k-2)$
G_2	3	$[\rho_1, \rho_2]$	$\lambda_1=\frac{1}{3}(2\rho_1+\rho_2)$, $\lambda_2=\frac{1}{3}(-\rho_1+\rho_2)$, $\lambda_3=\frac{1}{3}(-\rho_1-2\rho_2)$	$\frac{1}{3}(5, -1, -4)$	8
F_4	4	$(\rho_1, \rho_2, \rho_3, \rho_4)$	$\lambda_i=\rho_i$ for $i=1, 2, 3, 4$	$\frac{1}{2}(11, 5, 3, 1)$	18
E_6	8	$(\rho_1, \rho_2, \rho_3, \rho_4, \rho_5 : \rho)$	$\lambda_1=-\lambda_2=\rho/2$, $\lambda_i=\rho_{i-2}-\bar{\rho}$ for $i=3, 4, \ldots, 7$, $\lambda_8=-\bar{\rho}$ with $\bar{\rho}=(\rho_1+\rho_2+\ldots+\rho_5)/6$	$\frac{1}{2}(11, -11, 5, 3, 1, -1, -3, -5)$	24
E_7	8	$(\rho_1, \rho_2, \ldots, \rho_7)$	$\lambda_i=\bar{\rho}-\rho_{9-i}$, $i=2, \ldots, 8$ $\lambda_1=\bar{\rho}=(\rho_1+\rho_2+\ldots+\rho_7)/8$	$\frac{1}{4}(49, 5, 1, -3, -7, -11, -15, -19)$	36
E_8	8	$(\rho_1, \rho_2, \ldots, \rho_8)$	$\lambda_1=\bar{\rho}-\frac{4}{3}\rho_8$, $\lambda_i=\rho_{i-1}+\frac{4}{3}\rho_8-\bar{\rho}$, $i=2, 3, \ldots, 8$ with $\bar{\rho}=(\rho_1+\rho_2+\ldots+\rho_8)/6$	$(23, 6, 5, 4, 3, 2, 1, 0)$	60

which also satisfies the constraint $a_d(\boldsymbol{l}) = 0$ if $\boldsymbol{\lambda} = \boldsymbol{0}$. This function of $\boldsymbol{l}$ arises through the action of the Weyl group on $S_d(\boldsymbol{l})$. For the group E_6, the fact that the corresponding Weyl group contains only $\Sigma_2 \times \Sigma_6$ and not Σ_8 as a subgroup implies that the required basis is provided by the functions

$$P_m(\boldsymbol{l}) = \sum_{i=1}^{2} (l_i^m - \delta_i^m) \qquad \text{with } m = 1, 2$$

and

$$Q_m(\boldsymbol{l}) = \sum_{i=3}^{8} (l_i^m - \delta_i^m) \qquad \text{with } m = 1, 2, \ldots, 6; \tag{6.4}$$

however an alternative basis is preferable, making use of

$$P_m(\boldsymbol{l}) = \sum_{i=1}^{2} (l_i^m - \delta_i^m) \qquad \text{with } m = 1, 2$$

and

$$S_m(\boldsymbol{l}) = \sum_{i=1}^{8} (l_i^m - \delta_i^m) \qquad \text{with } m = 1, 2, \ldots, 8. \tag{6.5}$$

The results obtained will necessarily depend upon the labelling scheme used for the irreducible representations of the groups. This, in turn, depends upon the ordering relations used to define highest weights and indeed upon the specification of the weight and root spaces. The notation adopted is given in table 2, where, for each group, d and $\boldsymbol{\delta}$ are given, along with the scale factor N_ϕ and the relationship between the label $\boldsymbol{\lambda}$ and the more conventional irreducible representation labels $\boldsymbol{\rho}$ of Wybourne (1970) and Wybourne and Bowick (1977). From the data of table 2 it is then easy to determine $\boldsymbol{l}$, and hence to evaluate $C_2(\boldsymbol{l})$ since

$$C_2(\boldsymbol{l}) = \frac{1}{N_\phi} S_2(\boldsymbol{l}) = \frac{1}{N_\phi} \sum_{i=1}^{d} (l_i^2 - \delta_i^2). \tag{6.6}$$

Table 3. SU(N) expansion coefficients $\beta_p(k)$ where $(-4N)^p I_p^\omega = \Sigma_k \beta_p(k) L_k$.

I_p / L_k	I_2^*	I_3^*	I_4^*	I_5^*	I_6^*	I_7^*	I_8^*
$2^2 S_2$	1	N	4	$-2N^3+6N$	$-5N^4+10N^2+3$	$-9N^5+10N^3-15N$	$-14N^6+42N^2+4$
$2^3 S_3$		-1	$-2N$	$-2N^2-\frac{10}{3}$	$-\frac{40}{3}N$	$5N^4-30N^2-7$	$14N^5-\frac{140}{3}N^3-42$
$2^3 S_4$			1	$3N$	$5N^2+5N$	$5N^3+25N$	$70N^2+14$
$2^4 S_2^2$				-1	$-4N$	$-9N^2-4$	$-14N^3-24N$
$2^5 S_5$				-1	$-4N$	$-9N^2-7$	$-14N^3-42N$
$2^5 S_3 S_2$					2	$10N$	$28N^2+\frac{32}{3}$
$2^6 S_6$					1	$5N$	$14N^2+\frac{28}{3}$
$2^6 S_4 S_2$						-2	$-12N$
$2^6 S_3^2$						-1	$-6N$
$2^6 S_2^3$							$\frac{2}{3}$
$2^7 S_7$						-1	$-6N$
$2^7 S_5 S_2$							2
$2^7 S_4 S_3$							2
$2^8 S_8$							1

The dimensional formulae (3.1) appear explicitly for each group in question in, for example, the work of Dynkin (1957, p 358) and Wybourne (1974), or results may be gleaned for the classical groups either from an earlier tabulation by Wybourne (1970) or from the general formulae of El Samra and King (1979), and for the exceptional groups from the results of Wybourne and Bowick (1977) and Wybourne (1979). These last two papers are also extremely useful as a source of Kronecker products of representations of the exceptional groups. Their results were not sufficient however to give, for all the groups, the necessary η_p independent products even when $\boldsymbol{\mu}$ was taken to be the simplest representation $\boldsymbol{\omega}$. In these cases the list of products was extended by making use of Racah's formula (3.10) along with the modification rules (3.9), checked dimensionally. Alternatively, a related technique was used, involving products in the classical subgroup maximally embedded in each exceptional group and the modification rules appropriate to the exceptional group. In this way for $\boldsymbol{\mu} = \boldsymbol{\omega}$ sufficient products were evaluated to enable all the expansion coefficients $\beta_p(k)$ of (6.1) to be found for $p \leqslant 8$. The results are gathered together in tables 3–9.

As pointed out by Okubo (1977), the limitation to semi-simple Lie groups may be relaxed to cater for, in particular, the group U(N). Results appropriate to this group have been included for completeness in table 11 where, as in the table 10 for SU(N), an expansion has been made in terms of

$$T_m(l) = \sum_{i=1}^{N} [(l_i - c)^m - (\delta_i - c)^m] \tag{6.7}$$

Table 4. SO(N), Sp(N) expansion coefficients $\beta_p(k)$ (SO(N) upper sign, Sp(N) lower sign) where $[-4(N \mp 2)]^p I_p^\omega = \Sigma_k \beta_p(k) L_k$.

L_k \ I_p	I_2^*	I_3	I_4^*	I_5	I_6^*
$2^3 S_2$	1	$N \mp 2$	$\mp 2N + 4$	$-2N^3 \pm 4N^2 + 4N \mp 8$	$-5N^4 \pm 20N^3 - 20N^2 \mp 8N + 16$
$2^5 S_4$			1	$3N \mp 4$	$5N^2 \mp 14N + 14$
$2^6 S_2^2$				-1	$-4N \pm 5$
$2^7 S_6$					1
$2^8 S_4 S_2$					
$2^9 S_2^3$					
$2^9 S_8$					

L_k \ I_p	I_7	I_8^*
$2^3 S_2$	$-9N^5 \pm 50N^4 - 100N^3 \pm 72N^2 + 16N \mp 32$	$-14N^6 \pm 98N^5 - 280N^4 \pm 392N^3 - 224N^2 \mp 32N + 64$
$2^5 S_4$	$5N^3 \mp 24N^2 + 58N \mp 44$	$\mp 14N^3 + 112N^2 \mp 212N + 128$
$2^6 S_2^2$	$-9N^2 \pm 23N - 18$	$-14N^3 \pm 56N^2 - 94N$ [illegible] 56
$2^7 S_6$	$5N \mp 6$	$14N^2 \mp 34N + \frac{88}{3}$
$2^8 S_4 S_2$	-2	$-12N \pm 14$
$2^9 S_2^3$		$\frac{2}{3}$
$2^9 S_8$		1

Table 5. G_2 expansion coefficients $\beta_p(k)$ where $(-24)^p I_p^\omega = \Sigma_k \beta_p(k) L_k$.

L_k \ I_p	I_2^*	I_3	I_4	I_5	I_6^*	I_7	I_8
3^2S_2	2	12	84	660	2790	−31 746	−1038 678
$3^4S_2^2$			1	15	225/2	105/2	−35 705/2
3^6S_6					2	58	1062
$3^6S_2^3$						$-\frac{5}{2}$	−155
$3^8S_6S_2$							$\frac{8}{3}$
$3^8S_2^4$							$-\frac{5}{12}$

Table 6. F_4 expansion coefficients $\beta_p(k)$ where $24(-36)^p I_p^\omega = \Sigma_k \beta_p(k) L_k$.

L_k \ I_p	I_2^*	I_3	I_4	I_5	I_6^*	I_7	I_8^*
S_2	1	9	102	1380	25 275	583 608	14 373 636
2^2S_4					−195	−10 920	−403 865
S_2^2			4	50	1650	55 000	5226 490/3
2^4S_6					1	56	7448/3
$2^2S_4S_2$					−5	−280	−42 700/3
S_2^3					10	500	73 600/3
2^6S_8							$-\frac{5}{2}$
$2^4S_6S_2$							$\frac{56}{3}$
$2^4S_4^2$							$\frac{35}{6}$
$2^2S_4S_2^2$							−70
S_2^4							70

Table 7. E_6 expansion coefficients $\beta_p(k)$ with $(-144)^p I_p^\omega = \Sigma_k \beta_p(k) L_k$. Equation (6.5) defines S_k and P_2.

L_k \ I_p	I_2^*	I_3	I_4	I_5^*	I_6^*	I_7	I_8^*
6^2S_2	6	216	9936	552 096	27 598 968	694 836 144	−88 819 073 856
6^3S_3				−15 480	−2786 400	−347 185 440	−37 676 586 240
6^4S_4					42 120	9856 080	1450 282 320
$6^4S_2^2$			3	306	10 638	−1462 212	−349 499 448
6^5S_5				−12	−2160	−269 136	−29 206 656
$6^5S_3S_2$				10	1800	197 190	17 187 120
$6^5S_3P_2$				−20	−3600	−448 560	−48 677 760
6^6S_6					−24	−5616	−648 576
$6^6S_4S_2$					15	3510	454 500
$6^6S_3^2$					10	2340	303 000
$6^6S_2^3$						−99	−27 180
$6^7S_5S_2$						−21	−5544
$6^7S_3S_2^2$						$\frac{35}{2}$	4620
$6^7S_3S_2P_2$						−35	−9240
6^8S_8							−120
$6^8S_6S_2$							28
$6^8S_5S_3$							56
$6^8S_4^2$							35

Table 8. E_7 expansion coefficients $\beta_p(k)$ with $(-72)^p I_p^\omega = \Sigma_k \beta_p(k) L_k$.

$L_k \backslash I_p$	I_2^*	I_3	I_4	I_5	I_6^*	I_7	I_8^*
2^2S_2	12	216	504	143 712	8444 511	625 459 014	43 148 005 353
2^3S_3					52 920	64 562 400	5260 424 400
2^4S_4					23 940	2920 680	245 966 910
$2^4S_2^2$			6	312	4626	519 036	−58 368 524
2^5S_5							1481 760
2^6S_6					−48	−5856	−343 856
$2^6S_4S_2$					30	3660	242 840
$2^6S_3^2$					20	2440	485 680/3
$2^6S_2^3$						−108	−15 496
2^8S_8							−240
$2^8S_6S_2$							56
$2^8S_5S_3$							112
$2^8S_4^2$							70

Table 9. E_8 expansion coefficients $\beta_p(k)$ with $(-120)^p I_p^\omega = \Sigma_k \beta_p(k) L_k$.

$L_k \backslash I_p$	I_2^*	I_3	I_4	I_5	I_6	I_7	I_8^*
2^2S_2	60	1800	70 560	3398 400	193 628 160	12 726 835 200	830 257 643 520
2^4S_4							604 235 520
$2^4S_2^2$			36	3240	256 896	20 678 400	1642 493 568
2^6S_6							833 280
$2^6S_4S_2$							−1041 600
$2^6S_2^3$					30	5088	868 320
2^8S_8							−360
$2^8S_6S_2$							336
$2^8S_4^2$							210
$2^8S_4S_2^2$							−210
$2^8S_2^4$							105/2
2^8a_8							20 160

with $c = (N-1)/2$. This has the advantage of being such that, reverting to the usual notation $\{\rho\}$ for representations of both $U(N)$ and $SU(N)$, $T_m\{\rho\}$ is independent of N. Thus tables 10 and 11 indicate quite properly the fact that $I_p^\omega(\rho)$ is linear in N.

The N-dependence of the results for $SO(N)$ and $Sp(N)$ cannot be simplified by employing expansion functions (6.7) for differing values of c. However it should be stressed that the results expressed in table 4 are a considerable improvement on those tabulated by Nwachuku (1979) which involve functions S_m different from those used here, and which are such that S_3, for example, is not independent of S_2. It is these relationships between various S_m which must be used to demonstrate the result made obvious in table 4 that the independent invariants of $SO(N)$ and $Sp(N)$ are of order $2, 4, 6, \ldots$. Similarly Racah's listing of orders of the independent invariants given here in table 1 is readily confirmed from the results of tables 3–9 for $p \leqslant 8$.

Table 10. SU(N) expansion coefficients $\beta_p(k)$ where $(-4N)^p I_p^\omega = \Sigma_k \beta_p(k) L_k$. Equation (6.7) defines T_m.

L_k \ I_p	I_2	I_3	I_4	I_5	I_6	I_7	I_8
T_2	1	$-N+\frac{3}{2}$	$-\frac{3}{2}N+2$	$-2N+\frac{5}{2}$	$-\frac{5}{2}N+3$	$-3N+\frac{7}{2}$	$-\frac{7}{2}N+4$
T_3		-1	$N-2$	$2N-\frac{10}{3}$	$\frac{10}{3}N-5$	$5N-7$	$7N-\frac{28}{3}$
T_4			1	$-N+\frac{5}{2}$	$-\frac{5}{2}N+5$	$-5N+\frac{35}{4}$	$-\frac{35}{4}N+14$
T_2^2				$-\frac{1}{2}$	$\frac{1}{2}N-\frac{3}{2}$	$\frac{3}{2}N-\frac{25}{8}$	$\frac{25}{8}N-\frac{11}{2}$
T_5				-1	$N-3$	$3N-7$	$7N-14$
T_3T_2					1	$-N+\frac{7}{2}$	$-\frac{7}{2}N+\frac{25}{3}$
T_6					1	$N-\frac{7}{2}$	$-\frac{7}{2}N+\frac{28}{3}$
T_4T_2						-1	$N-4$
T_3^2						$-\frac{1}{2}$	$\frac{1}{2}N-2$
T_2^3							$\frac{1}{6}$
T_7						-1	$N-4$
T_5T_2							1
T_4T_3							1
T_8							1

Table 11. U(N) expansion coefficients $\beta_p(k)$ where $(-4N)^p I_p^\omega = \Sigma_k \beta_p(k) L_k$. Equation (6.7) defines T_m.

L_k \ I_p	I_2	I_3	I_4	I_5	I_6	I_7	I_8
T_1	$N-1$	$N-1$	$N-1$	$N-1$	$N-1$	$N-1$	$N-1$
T_2	1	$-N+\frac{3}{2}$	$-\frac{3}{2}N+2$	$-2N+\frac{5}{2}$	$-\frac{5}{2}N+3$	$-3N+\frac{7}{2}$	$-\frac{7}{2}N+4$
T_1^2		$-\frac{1}{2}$	$\frac{1}{2}N-1$	$N-\frac{3}{2}$	$\frac{3}{2}N-2$	$2N-\frac{5}{2}$	$\frac{5}{2}N-3$
T_3		-1	$N-2$	$2N-\frac{10}{3}$	$\frac{10}{3}N-5$	$5N-7$	$7N-\frac{28}{3}$
T_2T_1			1	$-N+\frac{5}{2}$	$-\frac{5}{2}N+\frac{9}{2}$	$-\frac{9}{2}N+7$	$-7N+10$
T_1^3				$-\frac{1}{6}$	$\frac{1}{6}N-\frac{1}{2}$	$\frac{1}{2}N-1$	$N-\frac{5}{3}$
T_4			1	$-N+\frac{5}{2}$	$-\frac{5}{2}N+5$	$-5N+\frac{35}{4}$	$-\frac{35}{4}N-14$
T_3T_1				-1	$N-3$	$3N-\frac{19}{3}$	$\frac{19}{3}N-\frac{34}{3}$
T_2^2				$-\frac{1}{2}$	$\frac{1}{2}N-\frac{3}{2}$	$\frac{3}{2}N-\frac{25}{8}$	$\frac{25}{8}N-\frac{11}{2}$
$T_2T_1^2$					$\frac{1}{2}$	$\frac{1}{2}N-\frac{7}{4}$	$-\frac{7}{4}N+4$
T_1^4						$-\frac{1}{24}$	$\frac{1}{24}N-\frac{1}{6}$
T_5				-1	$N-3$	$3N-7$	$7N-14$
T_4T_1					1	$-N+\frac{7}{2}$	$-\frac{7}{2}N+\frac{17}{2}$
T_3T_2					1	$-N+\frac{7}{2}$	$-\frac{7}{2}N+\frac{25}{3}$
$T_3T_1^2$						$-\frac{1}{2}$	$\frac{1}{2}N-2$
$T_2^2T_1$						$-\frac{1}{2}$	$\frac{1}{2}N-2$
$T_2T_1^3$							$\frac{1}{6}$
T_6					1	$-N+\frac{7}{2}$	$-\frac{7}{2}N+\frac{28}{3}$
T_5T_1						-1	$N-4$
T_4T_2						-1	$N-4$
$T_4T_1^2$							$\frac{1}{2}$
T_3^2						$-\frac{1}{2}$	$\frac{1}{2}N-2$
$T_3T_2T_1$							1
T_2^3							$\frac{1}{6}$
T_7						-1	$N-4$
T_6T_1							1
T_5T_2							1
T_4T_3							1
T_8							1

It is interesting to note that all the independent invariants of order $p \leq 8$ are found merely by considering the defining representation ω of each group, except in the case of SO($2N$). In this particular case the missing invariant of order $p = N$ is supplied by considering the irreducible spin representations, $\Delta_{\pm}$, whose generalised Casimir operator eigenvalues depend upon (6.3). This same dependence upon (6.3) is found also in the case of E_8 for the first independent Casimir operator beyond the second-order operator, namely I_8^{ω}, as indicated in table 9. In contrast to this, all independent invariants may not, in general, be found by considering the adjoint representation ϕ. This corresponds to the fact, pointed out by Racah (1951), that the eigenvalues of the operators C_p defined by (2.8) do not provide a complete set of labels for irreducible representations of the corresponding Lie group, failing to distinguish between a representation and its contragredient. In the case of the groups SU(N) this has been discussed in detail by Biedenharn (1963) who, as an alternative to the generalisation from (2.8) to (2.12) and the subsequent use of the defining representation ω, obtained a complete set of algebraically independent invariants through the consideration not only of the commutators (2.1) but also of the corresponding anticommutators.

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